

# ADJUNCTIONS BETWEEN BOOLEAN SPACES AND SKEW BOOLEAN ALGEBRAS

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**ABSTRACT.** We apply the representation theory of left-handed skew Boolean algebras by sections of their dual étale spaces, given in [7], to construct a series of dual adjunctions between the categories of locally compact Boolean spaces and left-handed skew Boolean algebras by means of extensions of certain enriched Hom-set functors induced by objects sitting in two categories. The constructed adjunctions are “deformations” of Stone duality obtained by the replacement in the latter of the category of Boolean algebras by the category of left-handed skew Boolean algebras. The constructions provide natural settings for the  $\omega$ -functor constructed in [12] and its left adjoint functor.

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## 1. INTRODUCTION

In [12] Leech and Spinks introduced a “twisted product”  $\omega$ -functor from the category of generalized Boolean algebras to the category of left-handed skew Boolean algebras. They outlined a proof based on the Freyd’s adjoint functor theorem that this functor has a left adjoint  $\Omega$  and described how the latter acts on finite left-handed skew Boolean algebras. In the present paper we observe that both  $\omega$  and  $\Omega$  can be elegantly given by extensions of enriched Hom-set functors composed with the functors establishing Stone duality. To be precise, we construct a series of adjunctions  $\Lambda_n \dashv \lambda_n$ ,  $n \geq 0$ , between the opposite category of the category of locally compact Boolean spaces and the category of left-handed skew Boolean algebras such that for each  $n \geq 0$  the adjunction  $\Lambda_n \dashv \lambda_n$  is induced by the object  $\{0, 1, \dots, n+1\}$  regarded as a primitive left-handed skew Boolean algebra or as a discrete topological space. This is a new series of examples of objects *sitting in two categories* or *schizophrenic objects*, see [14] and references therein for terminology, occurrence and significance of such objects. The adjunctions  $\Lambda_n \dashv \lambda_n$ ,  $n \geq 0$ , can be regarded as “deformations” of Stone duality if one replaces in the latter the category of Boolean algebras by the category of left-handed skew Boolean algebras. In the case  $n = 0$  the functor  $\lambda_0$  sends a locally compact Boolean space  $X$  to its dual generalized Boolean algebra  $X^*$ , and  $\Lambda_0$  is the functor sending a left-handed skew Boolean algebra  $S$  to the dual space  $(S/\mathcal{D})^*$  of  $S/\mathcal{D}$ . In the case  $n = 1$  the functor  $\lambda_1$  sends  $X$  to  $\omega(X^*)$  and  $\Lambda_1$  sends  $S$  to  $(\Omega(S))^*$ , where  $\omega$  and  $\Omega$  denote the Leech-Spinks functor and its left adjoint functor, respectively. To establish the results of this paper we rely on the technique, developed in [7], of representation of left-handed skew Boolean algebras by sections over compact clopen sets of their dual étale spaces, and homomorphisms of such algebras by cohomomorphisms of étale spaces.

The outline of this paper is as follows. In Section 2 we collect all necessary preliminaries. In Section 3 for each  $n \geq 0$  we construct functors  $\lambda_n$  and obtain the étale space representations of the skew Boolean algebras  $\lambda_n(X)$ . In Section 4 for every  $n \geq 0$  we construct functors  $\Lambda_n$ . We define the extended  $n$ -spectrum  $\Lambda_n(S)$  of a left-handed skew Boolean algebra  $S$  to be the set of all non-zero homomorphisms from  $S$  to the  $(n+2)$ -element primitive left-handed skew Boolean algebra. We topologize this set and obtain a locally compact Boolean space whose topology merges the locally compact Boolean space topology on  $(S/\mathcal{D})^*$  and the compact product topologies on  $\{1, \dots, n+1\}^{S_F^*}$ ,  $F \in (S/\mathcal{D})^*$ , via the sections of  $S^*$ . Then we prove that for each  $n \geq 0$  the functor  $\Lambda_n$  is the left adjoint to the functor  $\lambda_n$ . Finally, in Section 5 we describe the monads induced by the adjunctions  $\Lambda_n \dashv \lambda_n$ ,  $n \geq 0$ , and the Eilenberg-Moore categories of these monads. As a consequence we obtain that the adjunctions  $\Lambda_n \dashv \lambda_n$ ,  $n \geq 0$ , are monadic.

## 2. PRELIMINARIES

**2.1. The classical Stone duality.** A *generalized Boolean algebra* is a relatively complemented distributive lattice with the bottom (considered as a nullary operation). A homomorphism  $\varphi : B_1 \rightarrow B_2$  of generalized Boolean algebras is called *proper* [5], provided that for any  $b \in B_2$  there exists  $a \in B_1$  such that  $\varphi(a) \geq b$ . Let **GBA** denote the category of generalized Boolean algebras and proper homomorphisms. The category **BA** of Boolean algebras is a full subcategory of the category **GBA**. By **LCBS** we denote the category of locally compact Boolean spaces and continuous proper maps (recall that a map of topological spaces is *proper* if inverse images of compact sets are compact sets). The category **BS** of Boolean spaces is a full subcategory of the category **LCBS**.

Given a locally compact Boolean space  $X$ , all its compact clopen sets form a generalized Boolean algebra  $X^*$  called the *dual generalized Boolean algebra* of  $X$ . Equivalently,  $X^*$  is the generalized Boolean algebra of all continuous maps  $X \rightarrow \{0, 1\}$ , where  $\{0, 1\}$  is a discrete topological space, such that  $f^{-1}(1)$  is a compact set (such maps are called *vanished outside a compact set*). Let  $f \in \text{Hom}_{\text{LCBS}^{op}}(X_1, X_2)$ . Then  $(f^{op})^{-1}$  induces a morphism in  $\text{Hom}_{\text{GBA}}(X_1^*, X_2^*)$  which is denoted also by  $(f^{op})^{-1}$ . By  $\mathbf{A} : \text{LCBS}^{op} \rightarrow \text{GBA}$  we denote the functor given by  $\mathbf{A}(X) = X^*$ ,  $X \in \text{Ob}(\text{LCBS})$ , and  $\mathbf{A}(f) = (f^{op})^{-1}$ ,  $f \in \text{Hom}(\text{LCBS}^{op})$ . The restriction of  $\mathbf{A}$  to the category **BS** is the enriched Hom-set functor  $\text{Hom}_{\text{BS}}(-, \{0, 1\})$  because for  $X \in \text{Ob}(\text{BS})$  the Boolean algebra  $B^*$  consists of *all* continuous maps  $X \rightarrow \{0, 1\}$ .

The *spectrum*  $B^*$  of a generalized Boolean algebra  $B$  is the set of all prime filters of  $B$ , or, equivalently, the set of all non-zero homomorphisms from  $B$  to the two-element Boolean algebra  $\mathbf{2} = \{0, 1\}$ . One turns  $B^*$  into a topological space by proclaiming the sets  $M(a) = \{F \in B^* : a \in F\}$ ,  $a \in B$ , to form the base of the topology. Equipped with this topology  $B^*$  is a locally compact Boolean space called the *dual space* of  $B$ . Let  $f \in \text{Hom}_{\text{GBA}}(B_1, B_2)$ . Then  $f^{-1}$  induces a morphism in  $\text{Hom}_{\text{LCBS}}(B_2^*, B_1^*)$  which is denoted also by  $f^{-1}$ . By  $\mathbf{S} : \text{GBA} \rightarrow \text{LCBS}^{op}$  we denote the functor given via  $\mathbf{S}(B) = B^*$ ,  $B \in \text{Ob}(\text{GBA})$ , and  $\mathbf{S}(f) = (f^{-1})^{op}$ ,  $f \in \text{Hom}(\text{GBA})$ . The restriction of  $\mathbf{S}$  to the category **BA** is the enriched Hom-set functor  $\text{Hom}_{\text{BA}}(-, \mathbf{2})$  because given  $B \in \text{Ob}(\text{BA})$  the points of the space  $B^*$  are *all* Boolean algebra homomorphisms  $B \rightarrow \mathbf{2}$ .

*Stone duality for Boolean algebras* [15] (see also textbooks [4, 6]) states that the functors  $\text{Hom}_{\text{BS}}(-, \{0, 1\}) : \text{BS}^{op} \rightarrow \text{BA}$  and  $\text{Hom}_{\text{BA}}(-, \mathbf{2}) : \text{BA} \rightarrow \text{BS}^{op}$  establish an equivalence between the categories  $\text{BS}^{op}$  and **BA**. This duality is induced by the object  $\{0, 1\}$  considered as a Boolean algebra or as a discrete topological space.

*Stone duality for generalized Boolean algebras* [5, 15] is an extension of the above duality. It states that the functors  $\mathbf{A} : \text{LCBS}^{op} \rightarrow \text{GBA}$  and  $\mathbf{S} : \text{GBA} \rightarrow \text{LCBS}^{op}$

establish an equivalence between the categories  $\text{LCBS}^{op}$  and  $\text{GBA}$ . We call these functors *extensions* of enriched Hom-set functors bearing in mind that they extend the enriched Hom-set equivalence between  $\text{BS}^{op}$  and  $\text{BA}$ .

**2.2. Skew Boolean algebras.** Here we recall definitions and basic facts about skew Boolean algebras needed for our purposes. A detailed introduction to the theory of skew Boolean algebras can be found in [2, 9, 11]. A *skew lattice* is an algebra  $(S; \wedge, \vee)$  of type  $(2, 2)$  such that the operations  $\wedge$  and  $\vee$  are associative, idempotent and satisfy the absorption identities  $x \wedge (x \vee y) = x = x \vee (x \wedge y)$  and  $(y \vee x) \wedge x = x = (y \wedge x) \vee x$ . The *natural partial order*  $\leq$  on a skew lattice  $S$  is defined by  $x \leq y$  if and only if  $x \wedge y = y \wedge x = x$  or, equivalently,  $x \vee y = y \vee x = y$ . A skew lattice  $S$  is *symmetric* if  $x \wedge y = y \wedge x$  if and only if  $x \vee y = y \vee x$ . An element  $0$  of  $S$  is called a *zero* if  $x \wedge 0 = 0 \wedge x = 0$  for all  $x \in S$ .  $S$  is *Boolean* if it is symmetric, has a zero element and each principal subalgebra  $[x] = \{y \in S : y \leq x\} = x \wedge S \wedge x$  forms a Boolean lattice.

Let  $S$  be a Boolean skew lattice and  $x, y \in S$ . The *relative complement*  $x \setminus y$  is the complement of  $x \wedge y \wedge x$  in the Boolean lattice  $[x]$ . A *skew Boolean algebra* is a Boolean skew lattice, whose signature is enriched by the nullary operation  $0$  and the binary relative complement operation, that is, it is an algebra  $(S; \wedge, \vee, \setminus, 0)$ . Skew Boolean algebras satisfy distributivity laws  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  and  $(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x)$  [10, 2.5].

Let  $S$  be a skew lattice. It is called *rectangular* if there exist two sets  $L$  and  $R$  such that  $S = L \times R$ , and the operations  $\wedge$  and  $\vee$  are defined by  $(a, b) \wedge (c, d) = (a, d)$  and  $(a, b) \vee (c, d) = (c, b)$ . Let  $\mathcal{D}$  be the equivalence relation on  $S$  given by  $x \mathcal{D} y$  if and only if  $x \wedge y \wedge x = x$  and  $y \wedge x \wedge y = y$ . It is known [8, 1.7] that  $\mathcal{D}$  is a congruence relation, the  $\mathcal{D}$ -classes of  $S$  are maximal rectangular subalgebras and the quotient  $S/\mathcal{D}$  is the maximal lattice image of  $S$ . If  $S$  is a skew Boolean algebra then  $S/\mathcal{D}$  is the maximal generalized Boolean algebra image of  $S$  [2, 3.1].

A skew lattice is called *left-handed* (*right-handed*) if it satisfies the identities  $x \wedge y \wedge x = x \wedge y$  and  $x \vee y \vee x = y \vee x$  (respectively,  $x \wedge y \wedge x = y \wedge x$  and  $x \vee y \vee x = x \vee y$ ). In a left-handed skew Boolean algebra the rectangular subalgebras are *flat* in the sense that  $x \mathcal{D} y$  if and only if  $x \wedge y = x$  and  $y \wedge x = y$ .

A skew Boolean algebra  $S$  is called *primitive* if it has only one non-zero  $\mathcal{D}$ -class or, equivalently, if  $S/\mathcal{D}$  is the Boolean algebra  $\mathbf{2}$ . It is easy to see that finite primitive left-handed skew Boolean algebras are the skew Boolean algebras  $\mathbf{n} + \mathbf{2}$ ,  $n \geq 0$ . By the definition, the underlying set of  $\mathbf{n} + \mathbf{2}$  is the set  $\{0, 1, \dots, n+1\}$  and its non-zero  $\mathcal{D}$ -class is  $D = \{1, \dots, n+1\}$ , the operations on  $D$  being determined by left-handedness:  $x \wedge y = x$  and  $x \vee y = y$  for any  $x, y \in D$ .

Let  $\varphi : S_1 \rightarrow S_2$  be a homomorphism of skew Boolean algebras. Throughout the paper, by  $\overline{\varphi} : S_1/\mathcal{D} \rightarrow S_2/\mathcal{D}$  we denote the underlying homomorphism of generalized Boolean algebras. We call  $\varphi$  *proper* if  $\overline{\varphi}$  is a proper homomorphism of generalized Boolean algebras.

A skew Boolean algebra has *finite intersections* if any finite set of its elements has the greatest lower bound called the *intersection* with respect to the natural partial order. A skew Boolean algebra  $S$  with finite intersections considered as an algebra  $(S; \wedge, \vee, \setminus, \cap, 0)$ , where  $\cap$  is the binary operation of taking the intersection, is called a *skew Boolean  $\cap$ -algebra* [2].

By **Skew** we denote the category whose objects are left-handed skew Boolean algebras and whose morphisms are proper homomorphisms of skew Boolean algebras.

All skew Boolean algebras, considered in the sequel, are left-handed.

**2.3. Étale spaces.** Preliminaries on étale spaces can be found in any textbook on sheaf theory, e.g. in [3, 13]. An *étale space over  $X$*  is a triple  $(E, f, X)$ , where  $E$ ,

$X$  are topological spaces and  $f : E \rightarrow X$  is a surjective local homeomorphism. The points of  $E$  are called *germs*. If  $U$  is an open set in  $X$  then  $E(U)$  is the set of all *sections* of  $E$  over  $U$ . The *stalks* of  $E$  are the equivalence classes induced by  $f$ . For  $x \in X$  we denote the set of all  $y \in E$  such that  $f(y) = x$  by  $E_x$ . The set  $E_x$  is the stalk over  $x$ . If  $A \in E(U)$  then for  $x \in U$  by  $A(x)$  we denote the germ  $y \in A \cap E_x$ .

Let  $(\mathcal{A}, g, X)$  and  $(\mathcal{B}, h, Y)$  be étale spaces and  $f : X \rightarrow Y$  be a continuous map. An *f-cohomomorphism*  $k : \mathcal{B} \rightsquigarrow \mathcal{A}$  is a collection of maps  $k_x : \mathcal{B}_{f(x)} \rightarrow \mathcal{A}_x$  for each  $x \in X$  such that for every section  $s \in \mathcal{B}(U)$  the function  $x \mapsto k_x(s(f(x)))$  is a section of  $\mathcal{A}$  over  $f^{-1}(U)$ .

Let **Etale** be the category, whose objects are étale spaces over locally compact Boolean spaces and whose morphisms are étale space cohomomorphisms over continuous proper maps.

All étale spaces, considered in the sequel, are over locally compact Boolean spaces.

**2.4. Equivalence of the categories Skew and Etale.** Here we outline without proofs the results of [7], needed in the sequel.

First we construct the functor **SB** : **Etale**  $\rightarrow$  **Skew**. Let  $(E, \pi, X) \in \text{Ob}(\text{Etale})$ . Denote by  $E^*$  the set of all sections of  $E$  over compact clopen sets. Let  $U, V$  be compact clopen sets of  $X$  and  $A \in E(U)$ ,  $B \in E(V)$ . The *quasi-union*  $A \sqcup B$  of  $A$  and  $B$  is the section in  $E(U \cup V)$  given by

$$(A \sqcup B)(x) = \begin{cases} B(x), & \text{if } x \in V, \\ A(x), & \text{if } x \in U \setminus V. \end{cases}$$

The *quasi-intersection*  $A \sqcap B$  of  $A$  and  $B$  is the section in  $E(U \cap V)$  given by

$$(A \sqcap B)(x) = A(x) \text{ for all } x \in U \cap V.$$

The *quasi-complement*  $(A \setminus B)(x)$  is the section of in  $E(U \setminus V)$  given by

$$(A \setminus B)(x) = A(x) \text{ for all } x \in U \setminus V.$$

Equipped with these operations, together with the nullary operation  $\emptyset$  of specifying the empty section,  $E^* = (E^*, \sqcup, \sqcap, \setminus, \emptyset)$  is a left-handed skew Boolean algebra called the *dual skew Boolean algebra* to the étale space  $(E, \pi, X)$  and denoted by  $E^* = (E, f, X)^*$ .

Let  $(E, f, X), (G, g, Y) \in \text{Ob}(\text{Etale})$  and  $k \in \text{Hom}_{\text{Etale}}(E, G)$ . Since  $k$  preserves  $\sqcup, \sqcap$  and  $\emptyset$ , it can be considered as a skew Boolean algebra homomorphism from  $E^*$  to  $G^*$  which we also denote by  $k$ .

By **SB** we denote the functor from the category **Etale** to the category **Skew** given by **SB** $(E, f, X) = (E, f, X)^*$ ,  $(E, f, X) \in \text{Ob}(\text{Etale})$ , and **SB** $(k) = k$ ,  $k \in \text{Hom}(\text{Etale})$ .

Now we construct the functor **ES** : **Skew**  $\rightarrow$  **Etale**. Let  $S \in \text{Ob}(\text{Skew})$ . Throughout the paper, by  $\alpha : S \rightarrow S/\mathcal{D}$  we denote the canonical projection. We call a nonempty subset  $U$  of  $S$  a *filter* provided that:

- (1) for all  $a, b \in S : a \in U$  and  $b \geq a$  imply  $b \in U$ ;
- (2) for all  $a, b \in S : a \in U$  and  $b \in U$  imply  $a \wedge b \in U$ .

We call a subset  $U$  of  $S$  a *preprime filter* if  $U$  is a filter and  $\alpha(U)$  is a prime filter of  $S/\mathcal{D}$ . Let  $F$  be a prime filter of  $S/\mathcal{D}$ . Denote by  $\mathcal{PU}_F$  the set of all preprime filters of  $S$  contained in  $\alpha^{-1}(F)$ . Call minimal elements of the sets  $\mathcal{PU}_F$  *prime filters* of  $S$ . Prime filters of  $S$  are exactly minimal nonempty inverse images of 1 under the morphisms  $S \rightarrow \mathbf{3}$ .

Denote the set of all prime filters  $U$  of  $S$  such that  $\alpha(U) = F$  by  $\mathcal{U}_F$ . For any  $U_1, U_2 \in \mathcal{U}_F$  it holds either  $U_1 = U_2$  or  $U_1 \cap U_2 = \emptyset$ .

Let  $S \in \text{Ob}(\text{Skew})$ . Denote by  $S^*$  the set of all prime filters of  $S$ . We call  $S^*$  the *spectrum* of  $S$ . Let  $\pi : S^* \rightarrow (S/\mathcal{D})^*$  be the map given by

$$(1) \quad \pi(U) = F \text{ whenever } U \in \mathcal{U}_F.$$

It follows from the classical Stone duality that  $(S/\mathcal{D})^*$  is a locally compact Boolean space, whose base constitute the compact clopen sets

$$M'(A) = \{F \in (S/\mathcal{D})^* : A \in F\}, A \in S/\mathcal{D}.$$

For  $a \in S$  let

$$M(a) = \{F \in S^* : a \in F\}.$$

We topologize  $S^*$  so that the subbase of the topology is given by the sets  $M(a)$ ,  $a \in S$ . Then  $(S^*, \pi, (S/\mathcal{D})^*)$  is an étale space called the *dual étale space* of  $S$ . For  $F \in (S/\mathcal{D})^*$  the stalk  $S_F^*$  coincides with the set  $\mathcal{U}_F$ .

Let  $T, S \in \text{Ob}(\text{Skew})$  and  $k \in \text{Hom}_{\text{Skew}}(T, S)$ . From the classical Stone duality it follows that  $\bar{k}^{-1}$  induces a morphism in  $\text{Hom}_{\text{LCBS}}((S/\mathcal{D})^*, (T/\mathcal{D})^*)$ , which we denote also by  $\bar{k}^{-1}$ . Let  $F \in (S/\mathcal{D})^*$  and  $V \in S_F^* = \mathcal{U}_F$ . The set  $k^{-1}(V)$ , if non-empty, is some  $U' \in \mathcal{PU}_{\bar{k}^{-1}(F)}$ . We set  $\tilde{k}_F(U) = V$  provided that  $U \in T_{\bar{k}^{-1}(F)}^* = \mathcal{U}_{\bar{k}^{-1}(F)}$ ,  $k^{-1}(V) = U' \in \mathcal{PU}_{\bar{k}^{-1}(F)}$  and  $U \subset U'$ . In this way we have defined a map  $\tilde{k}_F : T_{\bar{k}^{-1}(F)}^* \rightarrow S_F^*$ . The collection of all  $\tilde{k}_F$ ,  $F \in (S/\mathcal{D})^*$ , forms a  $\bar{k}^{-1}$ -cohomomorphism  $\tilde{k} : T^* \rightarrow S^*$ .

We have constructed the functor  $\mathbf{ES} : \text{Skew} \rightarrow \text{Etale}$  given by  $\mathbf{ES}(S) = S^*$ ,  $S \in \text{Ob}(\text{Skew})$ , and  $\mathbf{ES}(k) = \tilde{k}$ ,  $k \in \text{Hom}(\text{Skew})$ .

**Theorem 1** ([7]). *The functors  $\mathbf{SB}$  and  $\mathbf{ES}$  establish an equivalence between the categories  $\text{Etale}$  and  $\text{Skew}$ , where the natural isomorphisms  $\beta : 1_{\text{Skew}} \rightarrow \mathbf{SB} \cdot \mathbf{ES}$  and  $\gamma : 1_{\text{Etale}} \rightarrow \mathbf{ES} \cdot \mathbf{SB}$  are given by*

$$(2) \quad \beta_S(a) = M(a), S \in \text{Ob}(\text{Skew}), a \in S;$$

$$(3) \quad \gamma_E(A) = N_A = \{N \in E^* : A \in N\}, E \in \text{Ob}(\text{Etale}), A \in E.$$

This theorem generalizes the classical Stone duality looked as an equivalence between the categories  $\text{LCBS}^{op}$  and  $\text{GBA}$ .

### 3. THE FUNCTORS $\lambda_n$ , $n \geq 0$

Let  $X \in \text{Ob}(\text{LCBS})$  and  $n \geq 0$ . We regard the set  $\{0, \dots, n+1\}$  as a discrete topological space. Denote by  $\lambda_n(X)$  the set of all continuous maps  $f$  from  $X$  to  $\{0, \dots, n+1\}$  such that  $f^{-1}(1), \dots, f^{-1}(n+1)$  are compact sets. Define the binary operations  $\wedge, \vee$  and the nullary operation  $0$  on  $\lambda_n(X)$  to be the induced operations of  $\wedge, \vee$  and  $0$  on the primitive left-handed skew Boolean algebra  $\mathbf{n} + \mathbf{2}$ . That is, for  $f, g \in \lambda_n(X)$  we put

$$(f \wedge g)(x) = f(x) \wedge g(x), (f \vee g)(x) = f(x) \vee g(x)$$

and we put the zero of  $\lambda_n(X)$  to be the zero function on  $X$ .

With respect to  $\wedge$  and  $\vee$  each  $\lambda_n(X)$  becomes a left-handed Boolean skew lattice. By adding to its signature the relative complement operation and the zero, we turn  $\lambda_n(X)$  into a left-handed skew Boolean algebra.

Now we generalize a standard idea of identifying characteristic functions  $X \rightarrow \{0, 1\}$  with subsets of  $X$  via the map  $f \mapsto f^{-1}(1)$ : for each  $n \geq 0$  we identify functions  $X \rightarrow \{0, \dots, n+1\}$  with ordered  $(n+1)$ -tuples of subsets of  $X$  via the bijection

$$f \mapsto (f^{-1}(\{1, \dots, n+1\}), f^{-1}(\{1, \dots, n\}), \dots, f^{-1}(\{1\})).$$

Applying the above identification, we have for every  $n \geq 0$

$$\lambda_n(X) = \{(A_{n+1}, \dots, A_1) : X \supseteq A_{n+1} \supseteq \dots \supseteq A_1, \\ A_i \text{ is compact and clopen for all } 1 \leq i \leq n+1\}.$$

**Lemma 2.** *Let  $n \geq 0$  and  $(A_i)_{n+1 \geq i \geq 1}, (B_i)_{n+1 \geq i \geq 1} \in \lambda_n(X)$ . Then*

$$(4) \quad (A_i)_{n+1 \geq i \geq 1} \wedge (B_i)_{n+1 \geq i \geq 1} = (A_i \cap B_{n+1})_{n+1 \geq i \geq 1}$$

$$(5) \quad (A_i)_{n+1 \geq i \geq 1} \vee (B_i)_{n+1 \geq i \geq 1} = ((A_i \setminus B_{n+1}) \cup B_i)_{n+1 \geq i \geq 1}.$$

*The zero of  $\lambda_n(X)$  corresponds to the  $(n+1)$ -tuple  $(\emptyset, \dots, \emptyset)$ .*

*Proof.* Let  $f : X \rightarrow \{0, \dots, n+1\}$  and  $g : X \rightarrow \{0, \dots, n+1\}$  correspond to  $(A_i)_{n+1 \geq i \geq 1}$  and  $(B_i)_{n+1 \geq i \geq 1}$ , respectively. Take the convention  $A_0 = B_0 = \emptyset$ . We have

$$f(A_i \setminus A_{i-1}) = i, \quad g(B_i \setminus B_{i-1}) = i, \quad 1 \leq i \leq n+1,$$

and

$$f(X \setminus A_1) = g(X \setminus B_1) = 0.$$

Applying  $(f \wedge g)(x) = f(x) \wedge g(x)$  we obtain

$$(f \wedge g)(x) = i \text{ if and only if } f(x) = i \text{ and } g(x) \neq 0, \quad 1 \leq i \leq n+1,$$

and

$$(f \wedge g)(x) = 0 \text{ if and only if } f(x) = 0 \text{ or } g(x) = 0.$$

Therefore,

$$(f \wedge g)((A_i \setminus A_{i-1}) \cap B_{n+1}) = i, \quad 1 \leq i \leq n+1,$$

and

$$(f \wedge g)((X \setminus A_{n+1}) \cup (X \setminus B_{n+1})) = 0.$$

This implies (4). The equality (5) is proved by a similar argument. The claim about the zero is clear since the zero corresponds to the zero function.  $\square$

We have constructed the object parts of the functors  $\lambda_n : \mathbf{LCBS}^{op} \rightarrow \mathbf{Skew}$ . Now we define their morphism parts. Let  $n \geq 0$  and  $f \in \mathbf{Hom}_{\mathbf{LCBS}^{op}}(X_1, X_2)$ . For  $(A_i)_{n+1 \geq i \geq 1} \in \lambda_n(X_1)$  we put

$$(6) \quad \lambda_n(f)((A_i)_{n+1 \geq i \geq 1}) = ((f^{op})^{-1}(A_i))_{n+1 \geq i \geq 1}.$$

This turns each  $\lambda_n$  into a functor from the category  $\mathbf{LCBS}^{op}$  to the category  $\mathbf{Skew}$ .

**Remark 3.** *Let  $\mathbf{S} : \mathbf{GBA} \rightarrow \mathbf{LCBS}^{op}$  be one of the functors establishing the classical Stone duality (see Subsection 2.1). Put  $\omega_n = \lambda_n \mathbf{S} : \mathbf{GBA} \rightarrow \mathbf{Skew}$ . Then the functor  $\omega_1$  coincides with (a version with proper morphisms of) the “twisted product” functor  $\omega$  introduced and studied in [12]. The above construction of  $\lambda_n$  via Hom-sets provides a natural setting for the functor  $\omega$  and its generalizations  $\omega_n$ .*

**Remark 4.** *Let  $\mathbf{SkewU}$  be the full subcategory of  $\mathbf{Skew}$  whose objects are all such  $S \in \mathbf{Ob}(\mathbf{Skew})$  that  $S/\mathcal{D}$  is a Boolean algebra and let  $n \geq 0$ . The restriction of the functor  $\lambda_n$  to the category  $\mathbf{BS}^{op}$  is the enriched Hom-set functor  $\mathbf{Hom}(\_, \{0, \dots, n+1\}) : \mathbf{BS}^{op} \rightarrow \mathbf{SkewU}$ . Therefore  $\lambda_n$  is an extension of an enriched Hom-set functor.*

**Remark 5.** *If  $n = 0$  then  $\lambda_0$  can be decomposed as  $\lambda_0 = \mathbf{iA}$ , where  $\mathbf{A} : \mathbf{LCBS}^{op} \rightarrow \mathbf{GBA}$  is one of the functors establishing the classical Stone duality (see Subsection 2.1) and  $\mathbf{i} : \mathbf{GBA} \rightarrow \mathbf{Skew}$  is the inclusion functor. Therefore  $\lambda_0$ , as well as all of the functors  $\lambda_n$ , can be regarded as “deformations” of the functor  $\mathbf{A}$ .*

Now we construct a geometric representation of  $\lambda_n(X)$ ,  $n \geq 0$ , as a sheaf of sections over  $X$ . Let the sets  $X_{(i)} = \{x_{(i)} : x \in X\}$ ,  $1 \leq i \leq n+1$ , be pairwise disjoint copies of  $X$ . We let  $Y = \bigcup X_{(i)}$  and think of the sets  $X_{(i)}$  as of layers of  $Y$ . Define the projection map  $\pi : Y \rightarrow X$  by putting  $\pi(x_{(i)}) = x$  for all  $1 \leq i \leq n+1$  and all  $x \in X$ . Now we topologize  $Y$ . We proclaim that a section  $s$  of  $Y$  belongs to the subbase of the topology  $\tau$  provided that  $\pi(s \cap X_{(i)})$  is compact and clopen in  $X$  for all  $i$ .

**Proposition 6.** *The map*

$$(7) \quad (A_i)_{n+1 \geq i \geq 1} \mapsto \{x_{(1)} : x \in A_1\} \cup \{x_{(2)} : x \in A_2 \setminus A_1\} \cup \cdots \cup \{x_{(n+1)} : x \in A_{n+1} \setminus A_n\}$$

*represents  $\lambda_n(X)$  as a skew Boolean algebras of sections over compact clopen sets of the sheaf of sections of  $Y$  with respect to the map  $\pi : Y \rightarrow X$ .*

*Proof.* The assignment (7) is clearly bijective. In addition, it is easy to see that it maps  $\wedge$  to  $\overline{\cap}$ ,  $\vee$  to  $\underline{\cup}$  and  $0$  to  $\emptyset$ .  $\square$

The following theorem shows that  $(Y, \pi, X)$  coincides with the étale space representation on  $\lambda_n(X)$ . Hence we obtain a geometric illustration for the étale space  $(\lambda_n(X))^*$ .

**Theorem 7.** *The triple  $(Y, \pi, X)$  is an étale space, and  $\lambda_n(X)$  is isomorphic to  $(Y, \pi, X)^*$  via the assignment (7)*

*Proof.* In view of Proposition 6, it remains to show only that  $\pi : Y \rightarrow X$  is a local homeomorphism. Let  $x \in Y$  and let  $V$  be a compact clopen neighborhood of  $\pi(x)$  in  $X^*$ . Assume  $x \in X_{(i)}$ . Then it is clear that  $\pi^{-1}(V) \cap X_{(i)}$  is an open neighborhood of  $x$  that is homeomorphic to  $V$ .  $\square$

**Corollary 8.** *Every stalk of  $(\lambda_n(X))^*$  has cardinality  $n+1$ .*

*Proof.* The statement follows from Theorem 7 since  $Y$  has  $(n+1)$ -element stalks by the construction.  $\square$

**Corollary 9.**  $\lambda_n(X)/\mathcal{D} \simeq X^*$ .

*Proof.* The statement follows from Theorem 7 and the fact recorded as [7, Lemma 6] that for an étale space  $(E, \pi, X)$  it holds  $(E, \pi, X)^* \simeq X^*$ .  $\square$

**Example 10.** *Let  $X$  be a non-empty set and  $Y$  be a finite non-empty set. Applying Proposition 6 and Theorem 7 it is easy to see that (a left-handed version of) the skew Boolean algebra  $P(X, Y)$  of partial functions from  $X$  to  $Y$  [11, Example 3.6 b)] is isomorphic to  $\lambda_{|Y|-1}(\mathbf{S}(\mathcal{P}(X)))$ .*

**Remark 11.** *Applying Proposition 6 it is easy to verify that the skew Boolean algebra  $\lambda_n(X)$  has finite intersections.*

From now on we fix the notation for the germs of the stalk  $(\lambda_n(X))^*_x, x \in X^*$ :

$$(8) \quad (\lambda_n(X))^*_x = \{x_{(1)}, \dots, x_{(n+1)}\},$$

where  $x_{(i)} = (\lambda_n(X))^*_x \cap X_{(i)}$  for all  $i$ . We think of  $x_{(i)}$  as of the germ over  $x$  belonging to the  $i$ -th layer.

The following lemma that will be needed in the sequel follows from (6) and Theorem 7.

**Lemma 12.** *Let  $f \in \text{Hom}_{\text{LCBS}^{op}}(X_1, X_2)$ . Then  $\overline{\lambda_n(f)}^{-1} = f^{op}$  for any  $x \in X_2$  and  $1 \leq i \leq n+1$  we have*

$$\widetilde{\lambda_n(f)}_x((f^{op}(x))_{(i)}) = x_{(i)}.$$

#### 4. THE FUNCTORS $\Lambda_n$ AND THE ADJUNCTIONS $\Lambda_n \dashv \lambda_n$ , $n \geq 0$

In the previous section we regarded  $\{0, \dots, n+1\}$  as an object of  $\text{LCBS}^{op}$ , which led us to the construction of a series of deformations  $\lambda_n$ ,  $n \geq 0$ , of the functor  $\mathbf{A}$ . In this section we regard  $\{0, \dots, n+1\}$  as an object  $\mathbf{n} + \mathbf{2}$  of  $\text{Skew}$  (the underlying set of  $\mathbf{n} + \mathbf{2}$  is  $\{0, \dots, n+1\}$ ) and consider the set of all non-zero homomorphisms from  $S$  to  $\mathbf{n} + \mathbf{2}$  in  $\text{Skew}$ . We shall introduce a topology on each of the obtained Hom-sets turning it into a locally compact Boolean space. This will lead us to a series of functors  $\Lambda_n : \text{Skew} \rightarrow \text{LCBS}^{op}$ ,  $n \geq 0$ , such that  $\Lambda_n$  is a deformation of the functor  $\mathbf{S}$  and is the left adjoint to the functor  $\lambda_n$ .

Throughout this section, we fix  $S$  to be a left-handed skew Boolean algebra,  $\alpha : S \rightarrow S/\mathcal{D}$  to be the canonical projection that maps  $a \in S$  to its  $\mathcal{D}$ -class, and  $\pi$  to be the projection of  $S^*$  onto  $(S/\mathcal{D})^*$  given by  $\pi(x) = F$  whenever  $x \in S_F^*$ , see (1). We also fix  $n \geq 0$  till the end of this section.

Denote by  $\Lambda_n(X)$  the set of all non-zero homomorphisms  $S \rightarrow \mathbf{n} + \mathbf{2}$  in  $\text{Skew}$ . We call  $\Lambda_n(X)$  the *extended  $n$ -spectrum* of  $S$ . Remark that the extended  $n$ -spectrum of  $S$  is not the same as the spectrum of  $S$  because the latter corresponds to the set of all homomorphisms  $\varphi : S \rightarrow \mathbf{n} + \mathbf{2}$  in  $\text{Skew}$  such that the inverse image  $\varphi^{-1}(1)$  is non-empty and minimal among the possible inverse images of 1.

The following lemma provides a convenient characterization of the elements of  $\Lambda_n(S)$ .

**Lemma 13.** *There is a bijective correspondence between elements of  $\Lambda_n(S)$  and functions  $f \in \{1, \dots, n+1\}^{S_F^*}$ , where  $F$  runs through  $(S/\mathcal{D})^*$ .*

*Proof.* Let  $g \in \Lambda_n(S)$ . By the classical Stone duality,  $\bar{g}^{-1}$  induces a continuous map from  $\mathbf{2}^*$  to  $(S/\mathcal{D})^*$ . Denote by  $G$  the only point of the space  $\mathbf{2}^*$ . Let  $F \in (S/\mathcal{D})^*$  be such that  $\bar{g}^{-1}(G) = F$ . Then  $\tilde{g}$  has the only one component  $\tilde{g}_G : S_F^* \rightarrow (\mathbf{n} + \mathbf{2})_G^* = \{1, \dots, n+1\}$ . This implies that the map  $g \mapsto \tilde{g}_G$  is a bijection.  $\square$

In what follows we identify elements of  $\Lambda_n(S)$  and elements of the set  $\cup_{F \in (S/\mathcal{D})^*} \{1, \dots, n+1\}^{S_F^*}$ .

We proceed to the construction of a topology on  $\Lambda_n(S)$ . The idea will be to merge the subspace topologies on  $\{1, \dots, n+1\}^{S_F^*}$ , that coincide with product topologies and therefore are compact by Tychonoff's Theorem, and the locally compact Boolean space topology on  $(S/\mathcal{D})^*$  via the sections of  $S^*$ . Let  $x \in S^*$  and let  $F = \pi(x)$ . For  $1 \leq i \leq n+1$  we put

$$p_i(x) = \{f \in \{1, \dots, n+1\}^{S_F^*} : f(x) = i\}.$$

Let  $s \in S$ . Recall that  $M(s) = \beta_S(s) = \{x \in S^* : s \in x\}$  is the section of  $S^*$  representing  $s$  as an element of  $S^{**}$  (see Subsection 2.4). For  $1 \leq i \leq n+1$  we put

$$L(s, i) = \bigcup_{x \in M(s)} p_i(x).$$

We turn  $\Lambda_n(S)$  into a topological space by proclaiming the sets  $L(s, i)$ ,  $s \in S$ ,  $1 \leq i \leq n+1$ , to form a subbase of the topology.

From now on we regard  $\Lambda_n(S)$  as a topological space with respect to the introduced topology. The subbase of the topology will refer to the introduced subbase, and the base of the topology will refer to the base of the topology, induced by this subbase.

It is immediate that the subspace topology of every  $\{1, \dots, n+1\}^{S_F^*}$  coincides with the product topology.

Assume  $f \in \{1, \dots, n+1\}^{S_F^*}$ . Call  $F$  the *projection* of  $f$  to  $(S/\mathcal{D})^*$  and denote it by  $\text{pr}(f)$ . It is easy to see that

$$\text{pr}(L(s, i)) = \pi(M(s)).$$



Therefore  $\text{pr}(L(s, i))$  is a compact clopen set of  $(S/\mathcal{D})^*$ .

We aim to show that  $\Lambda_n(S)$  is a locally compact Boolean space. For this, we shall make an insight into the structure of the topology on  $\Lambda_n(S)$  and its connection with the topology on  $(S/\mathcal{D})^*$  via the projection function.

Let  $A$  be a compact clopen subset of  $(S/\mathcal{D})^*$ . We put

$$G(A) = \{f \in \Lambda_n(S) : \text{pr}(f) \in A\}.$$

This definition implies that for any  $s \in S$  satisfying  $\pi(M(s)) = A$  we have

$$(9) \quad G(A) = \bigcup_{i=1}^{n+1} L(s, i).$$

**Lemma 14.** *The sets  $L(s, i)$  are clopen for all  $s \in S$  and all  $1 \leq i \leq n+1$ .*

*Proof.* Fix  $s \in S$  and  $1 \leq i \leq n+1$ . Since  $L(s, i)$  belongs to a subbase of a topology, it is open. Denote  $A = \pi(M(s))$ . Since  $A$  is clopen in  $(S/\mathcal{D})^*$  it follows that its complement  $A^c$  is clopen too. Hence  $A^c = \cup_{j \in J} A_j$  where  $A_j$  are some basic compact clopens of  $(S/\mathcal{D})^*$ . For every  $A_j$  we fix some  $s_j \in S$  such that  $\pi(M(s_j)) = A_j$ . Then the set

$$(L(s, i))^c = \left( \bigcup_{j \in J} \bigcup_{i=1}^{n+1} L(s_j, i) \right) \cup \left( \bigcup_{j=1, j \neq i}^{n+1} L(s, j) \right)$$

is open, and therefore  $L(s, i)$  is closed.  $\square$

In the following lemma we establish a property of projections of certain sets of  $\Lambda_n(S)$  needed in the sequel for the proof of Theorem 16.

**Lemma 15.** *Let  $A$  be a compact clopen set of  $(S/\mathcal{D})^*$ . Let  $k \geq 1$  and  $U_1, \dots, U_k \subseteq G(A)$  belong to the base of the topology on  $\Lambda_n(S)$ . We claim that the set*

$$\text{pr} \left( \bigcap_{j=1}^k (G(A) \setminus U_j) \right)$$

*is closed in  $(S/\mathcal{D})^*$ .*

*Proof.* Since all of the sets  $U_j$  are finite intersections of some of the sets  $L(s, i)$ , they are clopen by Lemma 14. It is clear that

$$\text{pr} \left( \bigcap_{j=1}^k (G(A) \setminus U_j) \right) = \text{pr} \left( G(A) \setminus \bigcup_{j=1}^k U_j \right) = A \setminus Z,$$

where  $Z$  is the set consisting of all those  $F \in (S/\mathcal{D})^*$  such that

$$\left( \bigcup_{j=1}^k U_j \right) \cap \{1, \dots, n+1\}^{S_F^*} = \{1, \dots, n+1\}^{S_F^*}.$$

It is enough to show that  $Z$  is open. We can assume that  $Z \neq \emptyset$  (otherwise  $Z$  is trivially open). Fix an arbitrary  $F \in Z$ . Let  $I$  be a subset of  $\{1, \dots, k\}$  such that each of the sets  $U_i$ ,  $i \in I$ , has a non-empty intersection with  $\{1, \dots, n+1\}^{S_F^*}$ . By  $B$  we denote the set of all  $s \in S$  such that  $s$  occurs in some  $L(s, i)$  used to form at least one of the sets  $U_i$ ,  $i \in I$ . Since  $I$  is finite and each  $U_i$  is a finite intersection of sets of the form  $L(s, i)$ , it follows that  $B$  is finite, too.

Since  $(S^*, \pi, (S/\mathcal{D})^*)$  is an étale space, and  $B$  is finite, there is a compact clopen set  $A$  in  $(S/\mathcal{D})^*$  such that  $A \subseteq \cap_{s \in B} \pi(M(s))$ ,  $F \in A$  and the restriction  $M(s)|_A$  of any  $s \in B$  to  $A$  is uniquely determined by the germ  $M(s)(F)$ . That is, if  $M(s)(F) = M(t)(F)$  for  $s, t \in B$  then  $M(s)|_A = M(t)|_A$ .

Let

$$P(F) = \{M(s)(F) : s \in B\}.$$

Since  $B$  is finite then  $P(F)$  is finite, too. Since

$$L(s, i) \cap \{1, \dots, n+1\}^{S_F^*} = p_i(s(F))$$

it follows that each of the sets  $U_i \cap \{1, \dots, n+1\}^{S_F^*}$ ,  $i \in I$ , is equal to some set

$$(10) \quad p_{j_1}(a_{i_1}) \cap \dots \cap p_{j_t}(a_{i_t}),$$

where  $\{a_{i_1}, \dots, a_{i_t}\} \subseteq P(F)$ .

Since

$$(11) \quad \{1, \dots, n+1\}^{S_F^*} = \left( \bigcup_{i \in I} U_i \right) \cap \{1, \dots, n+1\}^{S_F^*} = \bigcup_{i \in I} \left( U_i \cap \{1, \dots, n+1\}^{S_F^*} \right)$$

then the set  $\{1, \dots, n+1\}^{S_F^*}$  is a finite union of the sets of the form (10). Let  $G \in A$  and let

$$P(G) = \{s(G) : s \in B\}.$$

By the construction of  $A$ , we have that  $L(s, i) \cap \{1, \dots, n+1\}^{S_G^*}$  is non-empty and is equal to  $p_i(s(G))$  for all of the sets  $L(s, i)$ ,  $s \in B$ . It follows that the set

$$(12) \quad \left( \bigcup_{i \in I} U_i \right) \cap \{1, \dots, n+1\}^{S_G^*} = \bigcup_{i \in I} \left( U_i \cap \{1, \dots, n+1\}^{S_G^*} \right)$$

can be written as a finite union of the sets

$$p_{j_1}(b_{i_1}) \cap \dots \cap p_{j_t}(b_{i_t})$$

with  $\{b_{i_1}, \dots, b_{i_t}\} \subseteq P(G)$ .

Let  $g \in \{1, \dots, n+1\}^{S_G^*}$ . Consider an onto map  $\psi : P(F) \rightarrow P(G)$  given by  $\psi(M(s)(F)) = M(s)(G)$ ,  $s \in B$ . This map is well-defined since  $G \in A$  and  $M(s)|_A$  is uniquely determined by  $M(s)(F)$ . For every  $b \in P(G)$  fix some its inverse image  $b' \in P(F)$  with respect to  $\psi$ . Consider  $f \in \{1, \dots, n+1\}^{S_F^*}$  such that  $f(b') = g(b)$  for all  $b \in P(G)$ . Since (11) holds, it follows that either among the sets given by (10) there is one equal to

$$(13) \quad p_{f(b'_1)}(b'_1) \cap \dots \cap p_{f(b'_{i_s})}(b'_{i_s}),$$

where  $\{b'_1, \dots, b'_{i_s}\} \subseteq \{b' : b \in P(G)\}$  or among the sets given by (10) there are all the sets

$$C \cap p_{j_1}(c_1) \cap \dots \cap p_{j_m}(c_m),$$

where  $C$  is a set as given in (13),  $\{c_1, \dots, c_m\} \subseteq P(F) \setminus \{b' : b \in P(G)\}$  and  $j_1, \dots, j_m$  run through the set  $\{1, \dots, n+1\}$ .

But then, in any case, one of the sets  $U_i \cap \{1, \dots, n+1\}^{S_G^*}$  equals

$$p_{g(b_1)}(b_1) \cap \dots \cap p_{g(b_{i_s})}(b_{i_s}).$$

This proves that  $g$  belongs to the set (12). It follows that the set (12) equals the set  $\{1, \dots, n+1\}^{S_G^*}$ . Therefore  $G \in Z$  and thus  $A \subseteq Z$ . So  $Z$  is open, as required.  $\square$

**Theorem 16.**  $\Lambda_n(S)$  is a locally compact Boolean space.

*Proof.* We show first that  $\Lambda_n(S)$  is Hausdorff. Let  $f, g \in \Lambda_n(S)$  and  $f \neq g$ . Denote  $F = \text{pr}(f)$  and  $G = \text{pr}(g)$ . Assume  $F \neq G$ . Let  $A$  and  $B$  be disjoint compact clopen sets of  $(S/\mathcal{D})^*$  such that  $F \in A$  and  $G \in B$ . Consider some  $s, t \in S$  such that  $M(s) \in S^*(A)$  and  $M(t) \in S^*(B)$ . Put  $i = f(M(s)(F))$  and  $j = g(M(t)(G))$ . We have

$$f \in L(s, i), g \in L(t, j) \text{ and } L(s, i) \cap L(t, j) = \emptyset.$$

Assume now  $F = G$ . Let  $x \in S_F^*$  be such that  $f(x) \neq g(x)$ . Consider some  $s \in S$  such that  $x \in M(s)$ . In this case we have

$$f \in L(s, f(x)), g \in L(s, g(x)) \text{ and } L(s, f(x)) \cap L(s, g(x)) = \emptyset.$$

By Lemma 14 all of the sets  $L(s, i)$  are clopen.

We shall now show that all of the sets  $L(s, i)$  are compact. It is enough to show that all of the sets  $G(A)$ , where  $A$  runs through compact clopen subsets in  $(S/\mathcal{D})^*$ , are compact because every  $L(s, i)$  is a closed subset of some  $G(A)$ . Fix  $A$  to be a compact clopen set of  $(S/\mathcal{D})^*$  and  $s \in S$  to be such that  $\pi(M(s)) = A$ . Suppose that  $\{T_j\}_{j \in J}$  is a family of closed subsets of  $G(A)$  that has a finite intersection property. Each of the sets  $G(A) \setminus T_j$  is an open set. Therefore, there is a family  $\{X_{j,k}\}_{k \in K_j}$  of basic clopen sets contained in  $G(A)$  such that

$$G(A) \setminus T_j = \bigcup_{k \in K_j} X_{j,k}.$$

It follows that for each  $j \in J$  we have

$$T_j = G(A) \setminus \bigcup_{k \in K_j} X_{j,k} = \bigcap_{k \in K_j} (G(A) \setminus X_{j,k}).$$

The family

$$(14) \quad \{G(A) \setminus X_{j,k}\}_{k \in K_j, j \in J}$$

of all basic clopen sets contained in  $G(A)$  that are used to form the sets  $T_j$ ,  $j \in J$ , has a finite intersection property since the family  $\{T_j\}_{j \in J}$  has this property.

Let  $K$  be the closure of the family (14) with respect to taking finite intersections. That is,  $K$  is the family of sets containing the family (14) and all finite intersections of the sets belonging to (14). Clearly,  $K$  is closed with respect to finite intersections and is in fact the minimal such a family that contains (14). Since the family (14) has a finite intersection property, then so does  $K$ . By the construction, the elements of  $K$  are finite intersections  $\cap (G(A) \setminus X_i)$ , where  $X_i$  are some of the sets  $X_{j,k}$ ,  $k \in K_j, j \in J$ . By Lemma 15 the projections of all of the sets in  $K$  are closed in  $(S/\mathcal{D})^*$  and thus are closed in  $A$ . It is clear that  $\{\text{pr}(X) : X \in K\}$  has a finite intersection property because  $K$  has this property. So we have that the family  $\{\text{pr}(X) : X \in K\}$  is a family of closed subsets of  $A$  that has a finite intersection property. Applying the compactness of  $A$ , it follows that

$$(15) \quad \bigcap_{X \in K} \text{pr}(X) \neq \emptyset.$$

Fix some  $F \in \cap_{X \in K} \text{pr}(X)$  and for every  $X \in K$  let

$$A_X = X \cap \{1, \dots, n+1\}^{S_F^*}.$$

In view of (15) we have that the sets  $A_X$  are non-empty for all  $X \in K$ . Since every set in  $K$  is clopen, then every  $A_X$ ,  $X \in K$ , is clopen in  $\{1, \dots, n+1\}^{S_F^*}$  with respect to the subspace topology.

Show that the family  $\{A_X\}_{X \in K}$  has a finite intersection property. From the converse, assume that some its finite subfamily,  $\{A_X\}_{X \in L}$ , has an empty intersection.

Then we have

$$(16) \quad \emptyset = \bigcap_{X \in L} A_X = \bigcap_{X \in L} \left( X \cap \{1, \dots, n+1\}^{S_F^*} \right) = \left( \bigcap_{X \in L} X \right) \cap \{1, \dots, n+1\}^{S_F^*}.$$

The set  $\bigcap_{X \in L} X$  is a finite intersection of sets of  $K$  and thus also belongs to  $K$  since  $K$  is closed with respect to finite intersections. Because, according to (16), it holds  $(\bigcap_{X \in L} X) \cap \{1, \dots, n+1\}^{S_F^*} = \emptyset$ , we have  $F \notin \text{pr}(\bigcap_{X \in L} X)$ . The latter contradicts the choice of  $F$  in  $\bigcap_{X \in K} \text{pr}(X)$ . Therefore, the family  $\{A_X\}_{X \in K}$  has a finite intersection property.

Since  $\{1, \dots, n+1\}^{S_F^*}$  is compact and  $\{A_X\}_{X \in K}$  is a family of its closed subsets that has a finite intersection property, it follows that  $\bigcap_{X \in K} A_X \neq \emptyset$ . This implies  $\bigcap_{X \in K} X \neq \emptyset$ . Therefore  $\bigcap_{j \in J} T_j \neq \emptyset$ . It follows that  $G(A)$  is compact, as required.

We have proved that the space  $\Lambda_n(S)$  is Hausdorff and has a basis of compact clopens. It follows that  $\Lambda_n(S)$  is a locally compact Boolean space.  $\square$

**Remark 17.** *It is easy to deduce from the proof of Theorem 16 that  $\Lambda_n(S)$  is a Boolean space if and only if  $(S/\mathcal{D})^*$  is a Boolean space.*

Now we define the morphism part of the functor  $\Lambda_n : \text{Skew} \rightarrow \text{LCBS}^{op}$ . Let  $h \in \text{Hom}_{\text{Skew}}(S_1, S_2)$ . For  $f \in \Lambda_n(S_2)$  we set

$$(17) \quad (\Lambda_n(h))^{op}(f) = f \tilde{h}_{\text{pr}(f)}.$$

**Lemma 18.**  $\Lambda_n(h) \in \text{Hom}_{\text{LCBS}^{op}}(\Lambda_n(S_1), \Lambda_n(S_2))$ .

*Proof.* We shall show that  $(\Lambda_n(h))^{op} \in \text{Hom}_{\text{LCBS}}(\Lambda_n(S_2), \Lambda_n(S_1))$ . Let  $f \in \Lambda_n(S_2)$ . First we verify that  $(\Lambda_n(h))^{op}(f)$  is in  $\Lambda_n(S_1)$ . The component  $\tilde{h}_{\text{pr}(f)}$  of  $\tilde{h}$  is defined so that it maps  $(S_1^*)_{\tilde{h}^{-1}(\text{pr}(f))}$  to  $(S_2^*)_{\text{pr}(f)}$  (to recall the definition of  $\tilde{h}$  see Subsection 2.4). So  $(\Lambda_n(h))^{op}(f) = f \tilde{h}_{\text{pr}(f)}$  first maps  $(S_1^*)_{\tilde{h}^{-1}(\text{pr}(f))}$  to  $(S_2^*)_{\text{pr}(f)}$  via  $\tilde{h}_{\text{pr}(f)}$  and then maps  $(S_2^*)_{\text{pr}(f)}$  to  $\{1, \dots, n+1\}$  via  $f$ . It follows that  $(\Lambda_n(h))^{op}(f)$  belongs to  $\{1, \dots, n+1\}^{(S_1^*)_{\tilde{h}^{-1}(\text{pr}(f))}} \subseteq \Lambda_n(S_1)$ . We have shown even more:

$$\text{pr}((\Lambda_n(h))^{op}(f)) = \tilde{h}^{-1}(\text{pr}(f)).$$

We proceed to show that  $(\Lambda_n(h))^{op}$  is continuous. It is enough to verify that  $((\Lambda_n(h))^{op})^{-1}(L(s, i))$  is an open set for any  $s \in S_2$  and any  $1 \leq i \leq n+1$ . Based on the definition (17) it is easy to see that for any  $a \in S_1^*$  such that  $a$  belongs to the domain of some  $\tilde{h}_F$  we have

$$((\Lambda_n(h))^{op})^{-1}(p_i(a)) = p_i(\tilde{h}_F(a)).$$

The latter implies that

$$((\Lambda_n(h))^{op})^{-1}(L(s, i)) = L(h(s), i)$$

for any  $s \in S_2$  and any  $1 \leq i \leq n+1$ . This proves that  $(\Lambda_n(h))^{op}$  is continuous.

Finally, show that  $(\Lambda_n(h))^{op}$  is proper. It is enough to verify that  $((\Lambda_n(h))^{op})^{-1}$  regarded as a generalized Boolean algebra homomorphism  $(\Lambda_n(S_1))^* \rightarrow (\Lambda_n(S_2))^*$  is a proper homomorphism of generalized Boolean algebras. Let  $C \in (\Lambda_n(S_2))^*$  and consider some  $A \in (S_2/\mathcal{D})^*$  such that  $C \subseteq G(A)$ . Let  $s \in S_2$  be such that  $\pi(M(s)) = A$ . Since  $h$  is proper, there is  $t \in S_1$  such that  $\alpha(h(t)) \geq \alpha(s)$  in  $S_2/\mathcal{D}$  (here  $\alpha$  stands for the canonical projection  $S_2 \rightarrow S_2/\mathcal{D}$ ). Applying this and (9) we obtain

$$\begin{aligned} ((\Lambda_n(h))^{op})^{-1}(G(\pi(M(t)))) &= ((\Lambda_n(h))^{op})^{-1}\left(\bigcup_{i=1}^{n+1} L(t, i)\right) = \bigcup_{i=1}^{n+1} L(h(t), i) = \\ &G(\pi(M(h(t)))) \supseteq G(\pi(M(s))) = G(A) \supseteq C, \end{aligned}$$

proving that  $((\Lambda_n(h))^{op})^{-1}$  is proper.  $\square$

Now we are prepared to formulate and prove our adjunction theorem.

**Theorem 19.** *For each  $n \geq 0$  the functor  $\Lambda_n : \mathbf{Skew} \rightarrow \mathbf{LCBS}^{op}$  is the left adjoint to the functor  $\lambda_n : \mathbf{LCBS}^{op} \rightarrow \mathbf{Skew}$ . The unit of the adjunction  $\eta : 1_{\mathbf{Skew}} \rightarrow \lambda_n \Lambda_n$  is given by*

$$(18) \quad \eta_S(a) = \left( \bigcup_{i=1}^k L(a, i) \right)_{n+1 \geq k \geq 1}, S \in \mathbf{Ob}(\mathbf{Skew}), a \in S.$$

*Proof.* We fix  $S \in \mathbf{Ob}(\mathbf{Skew})$ ,  $X \in \mathbf{Ob}(\mathbf{LCBS}^{op})$  and  $\mu \in \mathbf{Hom}_{\mathbf{Skew}}(S, \lambda_n(X))$ . Our aim is to show that there is a unique  $u \in \mathbf{Hom}_{\mathbf{LCBS}^{op}}(\Lambda_n(S), X)$  such that  $\mu = \lambda_n(u)\eta_S$ .

The inverse image  $\bar{\mu}^{-1}$  of the homomorphism  $\bar{\mu} : S/\mathcal{D} \rightarrow X^*$ , that underlies  $\mu$ , induces a proper continuous map of locally compact Boolean spaces from  $X$  to  $(S/\mathcal{D})^*$  which we also denote by  $\bar{\mu}^{-1}$ .

Fix  $F \in X$ . We have that  $\tilde{\mu}_F$  maps  $S_{\bar{\mu}^{-1}(F)}^*$  to  $(\lambda_n(X))_F^*$ . Since the latter set is the  $(n+1)$ -element set  $\{F_{(1)}, \dots, F_{(n+1)}\}$  by (8), then  $\tilde{\mu}_F \in \{F_{(1)}, \dots, F_{(n+1)}\}^{S_{\bar{\mu}^{-1}(F)}^*}$ . It follows that  $\tilde{\mu}_F$  can be identified with a point of  $\Lambda_n(S)$  and we can regard that  $\tilde{\mu}_F \in \Lambda_n(S)$ .

We define the map  $u^{op} : X \rightarrow \Lambda_n(S)$  by the assignment  $F \mapsto \tilde{\mu}_F$ .

For  $s \in S$  and  $1 \leq i \leq n+1$  by  $\mu(s)_i$  we denote the set  $M(\mu(s)) \cap \{F_{(i)} : F \in X\}$ . Geometrically, this set consists of the points of  $M(\mu(s))$  belonging to the  $i$ th layer of  $(\lambda_n(X))^*$ . Applying the inverse map to the bijective map (7) it follows that the representation of  $\mu(s)$  as an  $(n+1)$ -tuple of subsets of  $X$  is the following:

$$(19) \quad \mu(s) = \left( \bigcup_{i=1}^k \pi(\mu(s)_i) \right)_{n+1 \geq k \geq 1}$$

It is clear that

$$(20) \quad \pi(M(\mu(s))) = \bigcup_{i=1}^{n+1} \pi(\mu(s)_i).$$

Verify that  $u^{op}$  is continuous. It follows from Theorem 7 and the construction of the topology on  $Y$  before that theorem that the sets  $\pi(\mu(s)_i)$  are compact and clopen subsets of  $X$  for all  $s \in S$  and  $1 \leq i \leq n+1$ . Let  $F \in X$  and  $x \in S_{\bar{\mu}^{-1}(F)}^*$ . It follows from the definition of  $u^{op}$  that for all  $1 \leq i \leq n+1$  we have

$$(u^{op})^{-1}(p_i(x)) = \begin{cases} F, & \text{if } \tilde{\mu}_F(x) = F_{(i)}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows that

$$(21) \quad (u^{op})^{-1}(L(s, i)) = \{F \in X : \tilde{\mu}_F(M(s)(F)) = F_{(i)}\} = \pi(\mu(s)_i),$$

which, as was just remarked, is an open set. This shows that  $u^{op}$  is continuous.

Now we verify that  $u^{op}$  is proper. For this we verify that  $(u^{op})^{-1}$  induces a proper homomorphism  $(\Lambda_n(S))^* \rightarrow X^*$  of generalized Boolean algebras, which we denote also by  $(u^{op})^{-1}$ . Since  $\mu$  is proper then so is  $\bar{\mu} : S/\mathcal{D} \rightarrow X^*$ . Let  $a \in X^*$ . Since  $\bar{\mu}$  is proper then there exists  $b \in S/\mathcal{D}$  such that  $\bar{\mu}(b) \geq a$ . Let  $s \in S$  be such that  $\pi(M(s)) = b$ . Applying (21) and (20) we have

$$(u^{op})^{-1}(G(b)) = (u^{op})^{-1} \left( \bigcup_{i=1}^{n+1} L(s, i) \right) = \bigcup_{i=1}^{n+1} \pi(M(\mu(s))_i) = \pi(M(\mu(s))) = \bar{\mu}(b) \geq a.$$

Therefore,  $u \in \mathbf{Hom}_{\mathbf{LCBS}^{op}}(\Lambda_n(S), X)$ .

Finally, we show that  $\mu = \lambda_n(u)\eta_S$ . For  $a \in S$  we have

$$\begin{aligned}
\lambda_n(u)(\eta_S(a)) &= \lambda_n(u) \left( \bigcup_{i=1}^k L(a, i) \right)_{n+1 \geq k \geq 1} && \text{(by (18))} \\
&= \left( (u^{op})^{-1} \left( \bigcup_{i=1}^k L(a, i) \right) \right)_{n+1 \geq k \geq 1} && \text{(by (6))} \\
&= \left( \bigcup_{i=1}^k \pi(\mu(a)_i) \right)_{n+1 \geq k \geq 1} && \text{(by (21))} \\
&= \mu(a) && \text{(by (19))},
\end{aligned}$$

as required.

The uniqueness of  $u$  is shown in a standard way, we leave the details to the reader.  $\square$

**Remark 20.** If  $n \geq 1$  then it is easy to verify that the map  $\eta_S$  is injective, and is therefore a faithful representation of  $S$  as a subalgebra of  $\lambda_n \Lambda_n(S)$ .

**Remark 21.** The restriction of the functor  $\Lambda_n$  to the category **SkewU** (see Remark 4) is an enriched Hom-set functor  $\text{Hom}_{\text{SkewU}}(-, \mathbf{n} + \mathbf{2})$ . This and Remark 4 show that the corresponding restrictions of the adjunctions  $\Lambda_n \dashv \lambda_n$ ,  $n \geq 0$ , are induced by the object  $\{0, \dots, n+1\}$  sitting in the categories **SkewU** and **BS**, in the same fashion as the classical Stone duality is induced by the object  $\{0, 1\}$  sitting in the categories **BA** and **BS**.

Recall that by  $\mathbf{A} : \text{LCBS}^{op} \rightarrow \text{GBA}$  and  $\mathbf{S} : \text{GBA} \rightarrow \text{LCBS}^{op}$  we denote the functors establishing the classical Stone duality, as given in Subsection 2.1. For  $n \geq 0$  we define the functors  $\Omega_n = \mathbf{A}\Lambda_n : \text{Skew} \rightarrow \text{GBA}$ . Recall (see Remark 3) that for  $n \geq 0$  by  $\omega_n$  we denote the functors  $\omega_n = \lambda_n \mathbf{S} : \text{GBA} \rightarrow \text{Skew}$ .

**Corollary 22.** Let  $n \geq 0$ . The functor  $\Omega_n : \text{Skew} \rightarrow \text{GBA}$  is the left adjoint to the functor  $\omega_n : \text{GBA} \rightarrow \text{Skew}$ . The unit of the adjunction  $\eta : 1_{\text{Skew}} \rightarrow \omega_n \Omega_n$  is given by  $\eta_S(a) = (\bigcup_{i=1}^k L(a, i))_{n+1 \geq k \geq 1}$ ,  $S \in \text{Ob}(\text{Skew})$ ,  $a \in S$ .

**Remark 23.** The functor  $\Omega_1$  coincides with (a version with proper morphisms of) the functor  $\Omega$  from [12], where there was outlined a proof that  $\Omega$  exists and the action of  $\Omega$  on finite objects of **Skew** was described. Corollary 22 provides a complete description of the action of  $\Omega$ . The definition of  $\Omega_1$  via Hom-sets provides a natural setting for  $\Omega$ .

## 5. THE ALGEBRAS OF THE MONADS OF THE ADJUNCTIONS $\Lambda_n \dashv \lambda_n$ , $n \geq 0$

All necessary preliminaries on monads can be found in [1, Chapter10]. Let  $n \geq 0$  be fixed throughout this section. We shall first describe the monad  $(T, \eta, \mu)$  over the category **Skew** that arises from the adjunction  $\Lambda_n \dashv \lambda_n$ . According to [1, Proposition 10.3] we have  $T = \lambda_n \Lambda_n$ ,  $\eta$  is the unit of the adjunction given by (18), and  $\mu = \lambda_n \epsilon_{\Lambda_n}$  is a natural transformation from  $T^2$  to  $T$ , where  $\epsilon : \Lambda_n \lambda_n \rightarrow 1_{\text{LCBS}^{op}}$  is the counit of the adjunction.

In this section it will be convenient to denote the points of the space  $\Lambda_n(S)$  by pairs  $(F, f)$ , where  $F$  runs through  $(S/\mathcal{D})^*$  and  $f$  runs through  $\{1, \dots, n+1\}^{S_F^*}$  for every  $F \in (S/\mathcal{D})^*$ . We introduce notation  $\mathcal{T}_{n+1}$  for the set of all transformations of the sets  $\{1, \dots, n+1\}$ . Suppose  $|S_F^*| = n+1$ . Denote  $S_F^* = \{F_{(1)}, \dots, F_{(n+1)}\}$ . For  $f \in \mathcal{T}_{n+1}$  by  $f_F$  we denote the function in  $\{1, \dots, n+1\}^{S_F^*}$  such that  $f_F(F_{(i)}) = f(i)$ .

**Lemma 24.** Let  $X \in \text{Ob}(\text{LCBS})$  and  $F \in X$ . There is a bijective correspondence between the points of the space  $\Lambda_n \lambda_n(X)$  and elements of the set  $X \times \mathcal{T}_{n+1}$ .

*Proof.* First, recall that  $(\lambda_n(X))/\mathcal{D} \simeq X^*$  by Corollary 9. Hence  $((\lambda_n(X))/\mathcal{D})^* \simeq X$ . The stalks  $(\lambda_n(X))_F^*$ ,  $F \in X$ , are  $(n+1)$ -element by Corollary 8. So for each  $F \in X$  there is a bijective correspondence between functions in  $\{1, \dots, n+1\}^{(\lambda_n(X))_F^*}$

and elements of  $\mathcal{T}_{n+1}$ . It follows there is a bijective correspondence between the points  $(F, f)$  of  $\Lambda_n \lambda_n(X)$  and the elements of the set  $X \times \mathcal{T}_{n+1}$ .  $\square$

In what follows we identify the points of  $\Lambda_n \lambda_n(X)$  with pairs  $(F, f_F)$ ,  $F \in X$ ,  $f \in \mathcal{T}_{n+1}$ . Let  $id$  denote the identical transformation in  $\mathcal{T}_{n+1}$ . We will sometimes write  $(F, f)$  for  $(F, f_F)$ .

In the following lemma we describe the counit of the adjunction  $\Lambda_n \dashv \lambda_n$ .

**Lemma 25.** *For  $X \in \text{Ob}(\text{LCBS})$  the map  $\epsilon_X \in \text{Hom}_{\text{LCBS}^{op}}(\Lambda_n \lambda_n(X), X)$ , such that  $\epsilon_X^{op}$  is given by  $F \mapsto (F, id)$ , is the component at  $X$  of the counit  $\epsilon$  of the adjunction  $\Lambda_n \dashv \lambda_n$ .*

*Proof.* Let  $X \in \text{Ob}(\text{LCBS})$ ,  $S \in \text{Ob}(\text{Skew})$  and  $h \in \text{Hom}_{\text{LCBS}^{op}}(\Lambda_n(S), X)$ . One has to show that there is a unique  $v \in \text{Hom}_{\text{Skew}}(S, \lambda_n(X))$  such that  $h = \epsilon_X \Lambda_n(v)$ . To construct the required  $v$  it is enough to construct  $\tilde{v} \in \text{Hom}_{\text{Etale}}(S^*, (\lambda_n(X))^*)$ . We have  $h^{op} \in \text{Hom}_{\text{LCBS}}(X, \Lambda_n(S))$ . Let  $F \in X$ . Denote  $h^{op}(F)$  by  $(G, f)$ , where  $G \in (S/\mathcal{D})^*$  and  $f \in \{1, \dots, n+1\}^{S_G^*}$ . We set  $\bar{v}^{-1}(F) = G$  and  $\tilde{v}_F(x) = F_{(f(x))}$  for all  $x \in S_G^*$ . The rest of the proof amounts to a routine verification that (i)  $\epsilon$  is a natural transformation; (ii)  $\tilde{v} \in \text{Hom}_{\text{Etale}}(S^*, (\lambda_n(X))^*)$  and (iii)  $v$  is unique such that the equality  $h = \epsilon_X \Lambda_n(v)$  holds. We leave this verification to the reader.  $\square$

We proceed to describe the action of the natural transformation  $\mu = \lambda_n \epsilon_{\Lambda_n}$  from  $T^2$  to  $T$ . Let  $S \in \text{Ob}(\text{Skew})$ . To describe  $\mu_S$  it is enough to describe  $\tilde{\mu}_S \in \text{Hom}_{\text{Etale}}((T^2(S))^*, (T(S))^*)$ . We shall obtain a geometric insight to the dual étale spaces of  $T^2(S) = \lambda_n \Lambda_n \lambda_n(S)$  and  $T(S) = \lambda_n \Lambda_n(S)$  based on (i) the construction of compact clopen sets of  $\Lambda_n(S)$  and (ii) the geometric structure of  $(\lambda_n(X))^*$  for any  $X \in \text{Ob}(\text{LCBS})$  which we know from Proposition 6 and Theorem 7. Applying Lemma 24 we identify the points of  $\Lambda_n \lambda_n \Lambda_n(S)$  with elements of the set  $\Lambda_n(S) \times \mathcal{T}_{n+1}$ . Since the points of the space  $\Lambda_n(S)$  are interpreted as pairs  $(F, f)$ , where  $F$  runs through  $(S/\mathcal{D})^*$  and  $f$  runs through  $\{1, \dots, n+1\}^{S_F^*}$ , it follows that the points of  $\Lambda_n \lambda_n \Lambda_n(S)$  can be identified with triples  $(F, f, g)$ , where  $F$  runs through  $(S/\mathcal{D})^*$ ,  $f$  runs through  $\{1, \dots, n+1\}^{S_F^*}$  and  $g \in \mathcal{T}_{n+1}$ . From now on we identify the points of  $\Lambda_n \lambda_n \Lambda_n(S)$  with the described triples.

From Lemma 25 we have

**Lemma 26.**  $\epsilon_{\Lambda_n(S)}^{op} : \Lambda_n(S) \rightarrow \Lambda_n \lambda_n \Lambda_n(S)$  is given by  $(F, f) \mapsto (F, f, id)$ .

**Lemma 27.**  $\tilde{\mu}_S$  is over  $\epsilon_{\Lambda_n(S)}^{op}$  and for every  $(F, f) \in \Lambda_n(S)$  the component  $\tilde{\mu}_{S(F,f)}$  is given by

$$(F, f, id)_{(i)} \mapsto (F, f)_{(i)}, \quad 1 \leq i \leq n+1.$$

*Proof.* The statement follows from Lemma 26 and (6) applying the correspondence (7).  $\square$

We are now prepared to characterize the algebras for the monad  $(T, \eta, \mu)$ . Recall [1, 10.3] that an *algebra of the monad*  $(T, \eta, \mu)$  or simply a *T-algebra* is a pair  $(S, \gamma)$  with  $S \in \text{Ob}(\text{Skew})$  and  $\gamma : T(S) \rightarrow S$  in  $\text{Hom}(\text{Skew})$  such that:

$$(22) \quad 1_S = \gamma \eta_S \quad \text{and} \quad \gamma \mu_S = \gamma T(\gamma).$$

A *morphism of T-algebras*  $h : (S, \gamma) \rightarrow (T, \delta)$  is a morphism  $h : S \rightarrow T$  in  $\text{Hom}(\text{Skew})$  such that  $h\gamma = \delta T(h)$ .

**Theorem 28.** (1) A pair  $(S, \gamma)$  with  $S \in \text{Ob}(\text{Skew})$  and  $\gamma : T(S) \rightarrow S$  in  $\text{Hom}(\text{Skew})$  is an algebra for the monad  $(T, \eta, \mu)$  if and only if  $S = \lambda_n(X)$  for some  $X \in \text{Ob}(\text{LCBS})$  and  $\gamma = \lambda_n \epsilon_X$ .

(2) A map  $h : \lambda_n(X_1) \rightarrow \lambda_n(X_2)$  is a morphism of T-algebras if and only if  $h = \lambda_n(f)$  for some  $f : X_1 \rightarrow X_2$  in  $\text{Hom}(\text{LCBS}^{op})$ .

*Proof.* Clearly, the equality  $1_S = \gamma\eta_S$  implies the equality  $1_{S/D} = \overline{\gamma}\overline{\eta_S}$  and the latter implies  $1_{(S/D)^*} = \overline{\eta_S}^{-1}\overline{\gamma}^{-1}$ . Let  $F \in (S/D)^*$ . Since in addition  $\overline{\eta_S}^{-1}$  maps  $(F, f) \in (T(S))^*$  to  $F$ , it follows that there is  $f \in \{1, \dots, n+1\}^{S_F^*}$  such that  $\overline{\gamma}^{-1}(F) = (F, f)$ . Now, the definition of  $\eta_S$  implies that  $\widetilde{\eta_{S(F,f)}}$  sends every  $x \in S_F^*$  to  $(F, f)_{(f(x))}$ . So for every  $x \in S_F^*$  it holds

$$(23) \quad x \xrightarrow{\widetilde{\eta_{S(F,f)}}} (F, f)_{(f(x))} \xrightarrow{\widetilde{\gamma_F}} x.$$

The latter shows that  $\widetilde{\eta_{S(F,f)}}$  is one-to-one and so is  $\widetilde{\gamma_F}$  restricted to the image of  $\widetilde{\eta_{S(F,f)}}$ . Since  $\widetilde{\eta_{S(F,f)}}$  is one-to-one then (23) implies that  $f$  must be one-to-one, too.

Our aim now is to show that  $f$  is a bijection. We know that  $\widetilde{\mu_S}$  acts as is given in Lemma 27 and  $\widetilde{\gamma_F}$  from act as is given in (23). So we have

$$(24) \quad (F, f, id)_{(f(x))} \xrightarrow{\widetilde{\mu_{S(F,f)}}} (F, f)_{(f(x))} \xrightarrow{\widetilde{\gamma_F}} x.$$

In view of the second equality of (22) it must hold  $(\widetilde{\gamma\mu_S})_F = (\widetilde{\gamma T(\gamma)})_F$ . Since  $(\widetilde{\gamma\mu_S})_F = \widetilde{\gamma_F}\widetilde{\mu_{S(F,f)}}$ , then  $(\widetilde{\gamma T(\gamma)})_F = \widetilde{\gamma_F}\widetilde{T(\gamma)}_{(F,f)}$ . This and (24) capture the action of  $\widetilde{T(\gamma)}_{(F,f)}$ : it must hold

$$(25) \quad (F, f, id)_{(f(x))} \xrightarrow{\widetilde{T(\gamma)}_{(F,f)}} (F, f)_{(f(x))}.$$

Now let us calculate the action of  $\widetilde{T(\gamma)}_{(F,f)}$  based on  $T = \lambda_n\Lambda_n$  and the known actions of  $\lambda_n$  and  $\Lambda_n$ . We have  $T(\gamma) = \lambda_n(\Lambda_n(\gamma))$ . We look first at  $\Lambda_n(\gamma)$ . For any  $F \in (S/D)^*$  and  $g \in \{1, \dots, n+1\}^{(S/D)^*_F}$  one has

$$(\Lambda_n(\gamma))^{op}(F, g) = (\overline{\gamma}^{-1}(F), g\widetilde{\gamma_F}) = (F, f, g\widetilde{\gamma_F}).$$

Now we look at  $T(\gamma) = \lambda_n(\Lambda_n(\gamma))$ . By Lemma 12 for any  $(F, f) \in \Lambda_n(S)$  and  $1 \leq i \leq n+1$  it holds:

$$(26) \quad (F, f, g\widetilde{\gamma_F})_{(i)} \xrightarrow{\widetilde{T(\gamma)}_{(F,f)}} (F, g)_{(i)}.$$

Since the action of  $\widetilde{T(\gamma)}_{(F,f)}$  can be given by either (25) or by (26), then the latter two expressions must be agreed. It follows that it must hold  $(F, f, id) = (F, f, g\widetilde{\gamma_F})$ , implying that  $id = f\widetilde{\gamma_F}$ . From the latter equality we obtain that  $\widetilde{\gamma_F}$  is one-to-one. Therefore, bearing in mind that  $f$  is one-to-one, it follows that  $f$  is a bijection. Hence  $|S_F^*| = n+1$ . Therefore, has all the stalks of  $S$  are  $(n+1)$ -element.

Let  $F \in (S/D)^*$  be fixed. Since  $|S_F^*| = n+1$  and  $f$  is a bijection, we can enumerate the germs of  $S_F^*$  so that  $S_F^* = \{F_{(1)}, \dots, F_{(n+1)}\}$  and  $f$  maps each  $F_{(i)}$  to  $i$ . Therefore,  $f = id_F$ . In this notation we have

$$(27) \quad \widetilde{\gamma_F}(F, id_F)_{(i)} = F_{(i)}.$$

We proceed to show that  $S \simeq \lambda_n((S/D)^*)$ . We can assume that  $(\lambda_n((S/D)^*))_F^* = S_F^* = \{F_{(1)}, \dots, F_{(n+1)}\}$  and then we have to show that  $S = \lambda_n((S/D)^*)$ . We first demonstrate the inclusion  $S \subseteq \lambda_n((S/D)^*)$ . Let  $1 \leq i \leq n+1$ . Consider the  $(n+1)$ -tuple  $C_i = (L(s, i), \dots, L(s, i)) \in T(S)$ . Let us calculate  $\widetilde{\gamma}(M(C_i))$ . Applying (7) and the definition of  $L(s, i)$  we have

$$M(C_i) = \{(F, f)_{(1)} : F \in (S/D)^* : f(M(s)(F)) = i\}.$$

This and (27) imply that

$$\widetilde{\gamma}(M(C_i)) = \{F_{(1)} : M(s)(F) = F_{(i)}\},$$



and therefore for all  $i = 1, \dots, n+1$  the set  $\{F_{(1)} : M(s)(F) = F_{(i)}\}$  is a section in  $S^*$ . It follows that  $s$  is such that the intersection of  $M(s)$  with each layer of  $S^*$  is again a section in  $S^*$ . This implies that  $M(s) \in (\lambda_n((S/\mathcal{D})^*))^*$ . Consequently,  $s \in \lambda_n((S/\mathcal{D})^*)$ , and thus  $S \subseteq \lambda_n((S/\mathcal{D})^*)$  is established.

Now we demonstrate the reverse inclusion  $\lambda_n(S/\mathcal{D}) \subseteq S$ . Let  $(A_i)_{n+1 \geq i \geq 1} \in \lambda_n((S/\mathcal{D})^*)$ . Let  $A_0 = \emptyset$ . For each  $j = 1, \dots, n+1$  define  $D_j \in T(S)$  by

$$D_j = (G(A_j \setminus A_{j-1}), \dots, G(A_j \setminus A_{j-1}), \emptyset, \dots, \emptyset),$$

with  $\emptyset$  occurring in the latter expression  $j-1$  times. Applying (27) one shows that

$$\gamma(D_j) = (A_j \setminus A_{j-1}, \dots, A_j \setminus A_{j-1}, \emptyset, \dots, \emptyset),$$

with  $\emptyset$  occurring  $j-1$  times as well. Since all  $\gamma(D_j)$  belong to  $S$  then their join that equals  $(A_i)_{n+1 \geq i \geq 1}$  also belongs to  $S$ . Therefore, the inclusion  $\lambda_n(S/\mathcal{D}) \subseteq S$  is established.

Now we verify the equality  $\gamma = \lambda_n \epsilon_X$ . From Lemma 25 we have the action of  $\epsilon_X$  and applying Lemma 12 we obtain that the action of  $\lambda_n \epsilon_X$  is given by  $\widetilde{\lambda_n \epsilon_X F}(F, id)_{(i)} = F_{(i)}$  that coincides with the action of  $\tilde{\gamma}_F$  given in (27).

We are left to prove the claim about morphisms. Assume first that  $f : X_1 \rightarrow X_2$  in  $\text{Hom}(\text{LCBS}^{op})$  and show that  $\lambda_n(f) : \lambda_n(X_1) \rightarrow \lambda_n(X_2)$  is a morphism of  $T$ -algebras from  $(T(\lambda_n(X_1)), \lambda_n \epsilon_{X_1})$  to  $(T(\lambda_n(X_2)), \lambda_n \epsilon_{X_2})$ . For every  $F \in X_2$  and  $i = 1, \dots, n+1$  applying Lemma 12 we have

$$\begin{aligned} (f^{op}(F), id_{f^{op}(F)})_{(i)} &\xrightarrow{\widetilde{\lambda_n \epsilon_{X_1}}_{f^{op}(F)}} (f^{op}(F))_{(i)} \xrightarrow{\widetilde{\lambda_n(f)}_F} F_{(i)}; \\ (f^{op}(F), id_{f^{op}(F)})_{(i)} &\xrightarrow{\widetilde{\lambda_n \Lambda_n(h)}_{(F, id_F)}} (F, id_F)_{(i)} \xrightarrow{\widetilde{\lambda_n \epsilon_{X_2 F}}} F_{(i)}. \end{aligned}$$

Therefore the required equality  $\lambda_n(f) \lambda_n \epsilon_{X_1} = \lambda_n \epsilon_{X_2} \lambda_n \Lambda_n \lambda_n(f)$  holds.

Assume now that  $h : \lambda_n(X_1) \rightarrow \lambda_n(X_2)$  is a morphism of  $T$ -algebras from  $(T(\lambda_n(X_1)), \lambda_n \epsilon_{X_1})$  to  $(T(\lambda_n(X_2)), \lambda_n \epsilon_{X_2})$ . We aim to show that  $h = \lambda_n((\bar{h}^{-1})^{op})$ . Applying (17) and Lemma 12 for any  $F \in X_2$ ,  $f \in \{1, \dots, n+1\}^{(\lambda_n(X_2))^*_F}$  and  $i = 1, \dots, n+1$  we obtain

$$(\bar{h}^{-1}(F), f \tilde{h}_F)_{(i)} \xrightarrow{\widetilde{\lambda_n \Lambda_n(h)}_{(F, f)}} (F, f)_{(i)}.$$

Since  $\lambda_n \epsilon_{X_2}$  is defined only on  $(F, id_F)_{(i)}$ ,  $F \in X_2$ ,  $1 \leq i \leq n+1$ , then the above map composed with  $\lambda_n \epsilon_{X_2}$  is defined only on  $(\bar{h}^{-1}(F), id_F \tilde{h}_F)_{(i)}$ ,  $F \in X_2$ ,  $1 \leq i \leq n+1$ . The action of this composition is given by

$$(\bar{h}^{-1}(F), id_F \tilde{h}_F)_{(i)} \xrightarrow{\widetilde{\lambda_n \Lambda_n(h)}_{(F, id_F)}} (F, id_F)_{(i)} \xrightarrow{\widetilde{\lambda_n \epsilon_{X_2 F}}} F_{(i)}.$$

This and the equality  $h \lambda_n \epsilon_{X_1} = \lambda_n \epsilon_{X_2} \lambda_n \Lambda_n(h)$  imply that  $\widetilde{\lambda_n \epsilon_{X_1}}$  is defined on all  $(\bar{h}^{-1}(F), id_F \tilde{h}_F)_{(i)}$ ,  $F \in X_2$ ,  $1 \leq i \leq n+1$ . Therefore,  $id_F \tilde{h}_F = id_{\bar{h}^{-1}(F)}$  and thus the action of  $\tilde{h}_F$  is given by  $(\bar{h}^{-1}(F))_{(i)} \mapsto F_{(i)}$ ,  $F \in X_2$ ,  $1 \leq i \leq n+1$ . It follows that  $h = \lambda_n((\bar{h}^{-1})^{op})$ . The proof is complete.  $\square$

Let  $\lambda_n(\text{LCBS}^{op})$  be the category whose objects are  $\lambda_n(X)$ ,  $X \in \text{Ob}(\text{LCBS})$  and whose arrows are  $\lambda_n(f)$ ,  $f \in \text{Hom}(\text{LCBS}^{op})$ .

**Corollary 29.** *The Eilenberg-Moore category of the monad  $(T, \eta, \mu)$  is isomorphic to the category  $\lambda_n(\text{LCBS}^{op})$ . Consequently, the adjunction  $\Lambda_n \dashv \lambda_n$  is monadic for every  $n \geq 0$ .*

*Proof.* The first statement follows from Theorem 28. The second statement holds because the category  $\lambda_n(\mathbf{LCBS}^{op})$  is obviously isomorphic to the category  $\mathbf{LCBS}^{op}$ .  $\square$

**Corollary 30.** *The category  $\lambda_n(\mathbf{LCBS}^{op})$  is a reflective subcategory of the category  $\mathbf{Skew}$ . The reflector is given by the functor  $\Lambda_n \lambda_n$ .*

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